Quantitative Economics / Faculty of Economics and Business
University of Amsterdam

Exam Dynamical Systems

Date: Tuesday January 15, 2008 Time: 14:00-17:00

- Write down your name and student ID on everything you hand in.
- This is *not* an open book exam.
- Motivate your answers. No motivation = no points.
- This exam consists of four questions.

Question 1:
Consider the following dynamical system:

$$\dot{x} = -x + 11y - 2z,$$
$$\dot{y} = -3x + y + 2z,$$
$$\dot{z} = 8y - 4z,$$

where $(x(0), y(0), z(0)) = (1, 1, 1)$ is given.

a) Compute the solution of this system. N.B. -4 is a root to a certain polynomial you might encounter.

b) Is the equilibrium of this system (asymptotically) stable?

Question 2:
Solve $t\dot{x}(t) + x(t) = \epsilon$, $\epsilon > 0$, for the initial condition $x(t_0) = x_0$, $t_0 \neq 0$.

Question 3:
Consider the following dynamical system:

$$\dot{x} = -y,$$
$$\dot{y} = \rho y + x^3 - x,$$

where $0 \leq \rho \leq 1$ a parameter. For the first part of this question $\rho = 0$.

a) Find all equilibria of the system, and determine their stability (if possible).

Hamiltonian systems (on the plane) have the following form:

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial y}, \\ \dot{y} &= -\frac{\partial H}{\partial x}, \end{aligned}$$

where $H : \mathbb{R}^2 \to \mathbb{R}$ is twice-differentiable.

b) Show that the system considered in this question is a Hamiltonian system (if $\rho = 0$) by finding the function H.

c) Show that each orbit of the system is contained in a single level set of H.

d) Sketch the phase portrait of the system.

e) Determine the stability of all points whose stability has not yet been decided.

From now on suppose that $\rho > 0$.

f) Find all equilibria of the system, and determine their stability.

g) Indicate how the phase portrait changes compared to the situation where $\rho = 0$.

Question 4:
Suppose there are n states. At time $t = 0, 1, \ldots$ the proportion of the system in state $i = 1, \ldots, n$ is p_{it}. The system is described at time t by the vector $p_t \in \Delta_n := \{x \in \mathbb{R}^n | \sum_{i=1}^n x_i = 1 \text{ and } x_i \geq 0 \text{ for all } i\}$. Note that the dimension of Δ_n is $n - 1$. The evolution of the system is the following: $p_{t+1} = Mp_t$, where $M \in \mathbb{R}^{n \times n}$ is the transition matrix. All entries of M are non-negative and the columns of M sum up to one, i.e. $M_{ij} \geq 0$ and $\sum_{j=1}^n M_{ij} = 1$. Let p^* denote a steady-state distribution of this system: $p^* \in \Delta_n$ and $p^* = Mp^*$. This is called a Markov process.

a) An invariant set of a dynamical system with flow $\Phi(t, x)$ is a set S such that for all $x_0 \in S$ and for all $t \in \mathbb{N}$, we have $\Phi(t, x_0) \in S$. Show that for linear discrete systems (i.e. $x_{t+1} = Ax_t$) this implies that for all $x \in S$, we have $Ax \in S$.

b) Show that Δ_n is an invariant set of the Markov process.

c) Show that a steady-state distribution p^* exists.

d) What can we conclude about the eigenvalues of M?