Question 1:

a) The eigenvalues and -vectors of the matrix

$$A = \begin{pmatrix} -1 & 11 & -2 \\ -3 & 1 & 2 \\ 0 & 8 & -4 \end{pmatrix}$$

are $\lambda_1 = 4\mathrm{i}$, $\lambda_2 = -4\mathrm{i}$ and $\lambda_3 = -4$ with corresponding eigenvectors $v_1 = (\frac{3}{2} - \mathrm{i}\frac{1}{2}, \frac{1}{2} + \mathrm{i}\frac{1}{2}, 1)$, $v_2 = (\frac{3}{2} + \mathrm{i}\frac{1}{2}, \frac{1}{2} - \mathrm{i}\frac{1}{2}, 1)$ and $v_3 = (2, 0, 3)$. The solution is

$$\mathbf{x}(t) = C^{-1} \begin{pmatrix} \cos 4t & -\sin 4t & 0 \\ \sin 4t & \cos 4t & 0 \\ 0 & 0 & -4 \end{pmatrix} C\mathbf{x}(0),$$

where $C = (\mathrm{Re}\ v_1, -\mathrm{Im}\ v_1, v_3)$. **[15 pt]**

b) Since $\mathrm{Re}\ \lambda_1 = \mathrm{Re}\ \lambda_2 = 0$ and $\lambda_3 < 0$, we see that the equilibrium of this system is stable but not asymptotically stable. **[5 pt]**

Question 2:
To solve this equation, first solve $t\dot{x} + x = 0$ using separation of variables:

$$\frac{\dot{x}}{x} = -\frac{1}{t} \Rightarrow \log x(t) - \log x(t_0) = \log t_0 - \log t \Rightarrow x(t) = \frac{x(t_0)t_0}{t}.$$

Suppose the solution of $t\dot{x}+x = \epsilon$ is $x(t) = c(t)u(t)$, where $c(t)$ is some unknown function and $u(t) = \frac{x(t_0)t_0}{t}$ (i.e. variation of constants). Then:

$$t[c'(t)u(t)+c(t)u'(t)]+c(t)u(t) = \epsilon \Rightarrow tc'(t)u(t) = \epsilon \Rightarrow c(t) = c(t_0)+\int_{t_0}^{t} \frac{\epsilon}{su(s)}\mathrm{d}s.$$

Since $c(t_0) = 1$, we see immediately that the solution is:

$$x(t) = \frac{x(t_0)t_0}{t} + \epsilon\left(1 - \frac{t_0}{t}\right).$$

[15 pt]

Question 3:

a) The three equilibria are $(0, 0)$, $(-1, 0)$ and $(1, 0)$. Linearizing the system, we get:

$$\dot{\mathbf{z}} = \begin{pmatrix} 0 & -1 \\ 3x^2 - 1 & 0 \end{pmatrix} \mathbf{z}.$$

It is easy to see that $(0, 0)$ is a saddle (eigenvalues ± 1). Since $(-1, 0)$ and $(1, 0)$ have pure complex eigenvalues $(\pm\mathrm{i}\sqrt{2})$, we cannot say anything about their stability. **[5 pt]**

b) This is straightforward integration: $H(x,y) = -\frac{1}{2}y^2 - \frac{1}{4}x^4 + \frac{1}{2}x^2$. **[3 pt]**

c) Since:

$$\frac{\partial H}{\partial t} = \frac{\partial H}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial H}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial H}{\partial x}\frac{\partial H}{\partial y} - \frac{\partial H}{\partial y}\frac{\partial H}{\partial x} = 0.$$

[3 pt]

d) We have to draw level sets of H:

$$-\frac{1}{2}y^2 - \frac{1}{4}x^4 + \frac{1}{2}x^2 = c \implies y = \pm\sqrt{-\frac{1}{2}x^4 + x^2 - c} \text{ or, } y = \pm\sqrt{f(x) - c}.$$

So what does $f(x)$ look like? Note that $f'(x) = -2x^3 + 2x$ and $f''(x) = -6x^2 + 2$, which implies a local minimum at $x = 0$ and global maxima at $x = 1$ and $x = -1$. The ellipses are determined by the x's for which $f(x) \geq c$. If $f(0) < c < \max_x f(x)$, then these are ellipses around $(-1, 0)$ and $(1, 0)$. At $c = f(0)$, the ellipses touch in $(0, 0)$ (a homoclinic connection; they don't have to know this term, but they should draw it correctly) and form an ∞-like loop. For $c < f(0)$, we get big loops around the ∞-like loop. **[10 pt]**

e) One can now see that $(-1, 0)$ and $(1, 0)$ are stable. **[4 pt]**

f) The equilibria stay the same. The linearized system becomes:

$$\dot{\mathbf{z}} = \begin{pmatrix} 0 & -1 \\ 3x^2 - 1 & \rho \end{pmatrix} \mathbf{z}.$$

Now $(0, 0)$ is still a saddle (the eigenvalues are $\rho/2 \pm \sqrt{\rho^2 + 4}/2$). The other two equilibria are now unstable with eigenvalues $\rho/2 \pm \sqrt{\rho^2 - 8}/2$, which are imaginary ($\rho \leq 1$) and have real part larger than zero. **[5 pt]**

g) The centers become sources and the homoclinic connection breaks. **[5 pt]**

Question 4:

a) Note that for a linearized system $\Phi(t, x_0) = A^t x_0$. If S is invariant, then for all $x_0 \in S$, $x_1 = Ax_0 \in S$. $x_2 = A^2 x_0 \in S$, etcetera. Now suppose that $x_t = A^t x_0 \in S$, then $A^{t+1}x_0 \in S$ or $A(A^t x_0) \in S$. If we pick $x = A^t x_0$, then we see that $x \in S$ should imply $Ax \in S$. **[5 pt]**

b) We have to show that for all $x \in \Delta_n$, $Mx \in \Delta_n$. This is easiest using matrix notation. The unit column vector will be denoted by ι. Since $x \in \Delta_n$, we know that $x \geq 0$ and $\iota' x = 1$. Since $M \geq 0$, it must be that $Mx \geq 0$. Next we have to show that $\iota'(Mx) = 1$. Note that by definition $\iota' M = \iota'$. Hence $\iota'(Mx) = (\iota' M)x = \iota' x = 1$. Since $Mx \geq 0$ and $\iota'(Mx) = 1$, we have that $Mx \in \Delta_n$. **[5 pt]**

c) Observe that $\iota' M = \iota'$. Hence $M'\iota = \iota$. Therefore $\lambda^* = 1$ is an eigenvalue of M'. But an eigenvalue of M' is also an eigenvalue of M. Therefore there exists a (eigen)vector v such that $v = Mv$. By appropriate scaling, p^* is this vector. **[5 pt]**

d) Since, Δ_n is invariant, we know that the equilibrium p^* is not unstable (otherwise the linearity of the system would imply that solutions go to infinity, which contradicts the invariancy of Δ_n). Hence all eigenvalues of M are in absolute value smaller than or equal to one. And (at least) one eigenvalue is equal to one. [**5 pt**]