Solution to exam dynamical systems June 25, 2008

Question 1:
The eigenvalues and eigenvector of A are

$$\lambda = 3\frac{1}{2} + \mathrm{i}\frac{1}{2}\sqrt{3} \text{ and } w = (1 - \mathrm{i}\sqrt{3}, 2)$$

as well as their conjugates. Note that $r = \sqrt{13}$ and $\omega = \frac{1}{3}\pi$. Then take $C = [\mathrm{Re}\, w - \mathrm{Im}\, w]$. The solution is given by:

$$x(t) = C \begin{bmatrix} \sqrt{13}^t \cos\frac{\pi t}{3} & -\sqrt{13}^t \sin\frac{\pi t}{3} \\ \sqrt{13}^t \sin\frac{\pi t}{3} & \sqrt{13}^t \cos\frac{\pi t}{3} \end{bmatrix} C^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Question 2:

a) At fixed points, we have $\dot{x} = \dot{y} = 0$. There are two fixed points: one at $(0,0)$ and one at $(4,2)$. To determine the stability we need to examine the eigenvalues of the Jacobian at these points. The Jacobian is given by:

$$DF(x,y) = \begin{bmatrix} \frac{1}{4}x & -1 \\ -1 & 2y \end{bmatrix}$$

The eigenvalues of $DF(0,0)$ are 1 and -1. Hence $(0,0)$ is a saddle. The eigenvalues of $DF(4,2)$ are $\frac{1}{2}(5+\sqrt{13}) > 0$ and $\frac{1}{2}(5-\sqrt{13}) > 0$. Hence $(0,0)$ is unstable.

b) The new fixed points are $(0,0,G(0,0))$ and $(4,2,G(4,2))$. The Jacobian becomes

$$\begin{bmatrix} \frac{1}{4}x & -1 & 0 \\ -1 & 2y & 0 \\ G_x & G_y & -1 \end{bmatrix},$$

where G_x and G_y the partial derivatives of G to x and y respectively. The first two eigenvalues of this matrix are the eigenvalues found in the previous exercise. The third eigenvalue is -1. Hence the new fixed points are all saddles.

Question 3:

a) See p. 4 of the syllabus. A flow satisfies the following two criteria: $\Phi(0,x) = x$ and $\Phi(t_1+t_2,x) = \Phi(t_2,\Phi(t_1,x))$.

b) Observe that:

$$\Phi(0,x) = \frac{x}{x+(1-x)} = x.$$

Moreover:

$$\begin{aligned}\Phi(t_2, \Phi(t_1, x)) &= \frac{\Phi(t_1, x)}{\Phi(t_1, x) + (1 - \Phi(t_1, x))\mathrm{e}^{t_2}} \\ &= \frac{\frac{x}{x+(1-x)\mathrm{e}^{t_1}}}{\frac{x}{x+(1-x)\mathrm{e}^{t_1}} + \frac{(1-x)\mathrm{e}^{t_1}}{x+(1-x)\mathrm{e}^{t_1}}\mathrm{e}^{t_2}} \\ &= \frac{x}{x + (1-x)\mathrm{e}^{t_1+t_2}} = \Phi(t_1 + t_2, x)\end{aligned}$$

c) Since $x(t) = \Phi(t, x_0)$, we have $\dot{x} = \frac{\partial \Phi}{\partial t}$. Hence $f(x) = \frac{\partial \Phi}{\partial t}$. Note that:

$$\begin{aligned}\frac{\partial}{\partial t}\Phi(t, x_0) &= \frac{-x_0(1-x_0)\mathrm{e}^t}{(x_0 + (1-x_0)\mathrm{e}^t)^2} \\ &= \frac{x_0}{x_0 + (1-x_0)\mathrm{e}^t} \times \frac{-(1-x_0)\mathrm{e}^t}{x_0 + (1-x_0)\mathrm{e}^t} \\ &= x(t)\left(\frac{x_0}{x_0 + (1-x_0)\mathrm{e}^t} - \frac{x_0 + (1-x_0)\mathrm{e}^t}{x_0 + (1-x_0)\mathrm{e}^t}\right) \\ &= x(t)(x(t) - 1).\end{aligned}$$

Hence $f(x) = x(x-1)$.

Question 4:

a) True. Note that:

$$\frac{x_{n+1}}{x_n} = 1 + \frac{x_n - 1}{x_n} \implies r_n = 1 + \frac{1}{r_{n-1}}$$

and $\frac{1}{2} + \frac{1}{2}\sqrt{5}$ is a fixed point of this difference equation. Our sequence starts at $r_0 = 1$. The question is whether it converges to $\frac{1}{2} + \frac{1}{2}\sqrt{5}$. Note that $\frac{1}{2} + \frac{1}{2}\sqrt{5} > r_n > r_{n-1}$. This only shows that $\lim_{n\to\infty} r_n \leq \frac{1}{2} + \frac{1}{2}\sqrt{5}$, but it is close enough as a proof. (Alternatively: rewrite the system as a first-order linear difference equation, solve and calculate the limit.)

b) False. The function is not Lipschitz-continuous at $x = 0$.

c) True. Linearizing the function around a fixed point shows that the Jacobian is the Hessian of F. Since F is strictly concave, the eigenvalues of the Hessian are strictly negative and hence the fixed point is asymptotically stable.