Oplossingen Tentamen Kres 3, 14 januari 2011

1.

$$f_{X,Y,Z}(x,y,z) = \begin{cases} c(x+y)\mathrm{e}^{-x-y-z}, & \text{voor } x,y,z>0 \\ 0, & \text{elders} \end{cases}$$

(a)

$$\begin{aligned} 1 &= c\int_0^\infty\int_0^\infty\int_0^\infty (x+y)\mathrm{e}^{-x-y-z}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= c\int_0^\infty\int_0^\infty (x+y)\mathrm{e}^{-x-y}\,\mathrm{d}x\,\mathrm{d}y \qquad \text{de integraal over } z \text{ is 1 (EXP(1) pdf)} \\ &= 2c\int_0^\infty\int_0^\infty x\mathrm{e}^{-x-y}\,\mathrm{d}x\,\mathrm{d}y \qquad \text{wegens x,y symmetrie} \\ &= 2c\int_0^\infty x\mathrm{e}^{-x}\,\mathrm{d}x \qquad \text{de integraal is de verwachting van een EXP(1) stochast} \\ &= 2c \end{aligned}$$

Dit geeft $c = \frac{1}{2}$

$$f_{X,Y}(x,y) = \frac{1}{2}\int_0^\infty (x+y)\mathrm{e}^{-x-y-z}\,\mathrm{d}z = \frac{1}{2}(x+y)\mathrm{e}^{-x-y}, \qquad (x,y>0)$$

$$\begin{aligned} f_Y(y) &= \frac{1}{2}\int_0^\infty (x+y)\mathrm{e}^{-x-y}\,\mathrm{d}x \\ &= \frac{1}{2}\int_0^\infty x\mathrm{e}^{-x-y}\,\mathrm{d}x + \frac{1}{2}\int_0^\infty y\mathrm{e}^{-x-y}\,\mathrm{d}x \\ &= \frac{1}{2}(1+y)\mathrm{e}^{-y} \qquad (y>0) \end{aligned}$$

(b) Wegens x, y symmetrie hebben X en Y dezelfde pdf,

$$f_X(x) = \frac{1}{2}(1+x)\mathrm{e}^{-x} \qquad (x>0)$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{(x+y)\mathrm{e}^{-x-y}}{(1+x)\mathrm{e}^{-x}} = \frac{(x+y)\mathrm{e}^{-y}}{(1+x)}, \qquad (y>0)$$

$$\begin{aligned} E[Y|X=x] &= \int_0^\infty \frac{y(x+y)\mathrm{e}^{-y}}{(1+x)}\,\mathrm{d}y \\ &= \frac{x}{1+x}\int_0^\infty y\mathrm{e}^{-y}\,\mathrm{d}y + \frac{1}{1+x}\int_0^\infty y^2\mathrm{e}^{-y}\,\mathrm{d}y \\ &= \frac{x+2}{x+1} \end{aligned}$$

De covariantie:

$$E[XY] = E[XE[Y|X]] = \int_0^\infty x\left(\frac{x+2}{x+1}\right)\frac{1}{2}(1+x)e^{-x}\mathrm{d}x = \frac{1}{2}\int_0^\infty (x^2+2x)e^{-x}\,\mathrm{d}x = 2$$

of direct:

$$\begin{aligned} E(XY) &= \frac{1}{2}\int_0^\infty \int_0^\infty xy(x+y)e^{-x-y}\,\mathrm{d}y\,\mathrm{d}x \\ &= \frac{1}{2}\int_0^\infty xe^{-x}\int_0^\infty y(x+y)e^{-y}\,\mathrm{d}y\,\mathrm{d}x \\ &= \frac{1}{2}\int_0^\infty xe^{-x}(x+2)\,\mathrm{d}x \\ &= \frac{1}{2}\int_0^\infty (x^2+2x)e^{-x}\,\mathrm{d}x = \frac{2+2}{2} = 2. \end{aligned}$$

$$E[X] = E[Y] = \frac{1}{2}\int_0^\infty y(1+y)e^{-y}\,\mathrm{d}y = \frac{1}{2}+1 = 1\frac{1}{2}$$

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y] = 2 - (1\frac{1}{2})^2 = -\frac{1}{4}$$

(c) Bepaal de joint pdf van $V = X - Y$ en $W = X + Y$. Zijn V en W s.o? $V = X - Y$, $W = X + Y \Rightarrow X = (V+W)/2$, $Y = (W-V)/2$.

$$J = \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad ||J|| = |\frac{1}{2}| = \frac{1}{2}$$

$$f_{V,W}(v,w) = \frac{1}{4}w\mathrm{e}^{-w}, \qquad v+w>0,\ w>v$$

V en W zijn niet s.o. (drager)

2. (a) $V = X + 2Y$ en $W = X - 2Y$, zodat $X = (V+W)/2$, $Y = (V-W)/4$

$$J = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{-1}{4} \end{pmatrix} \qquad |J| = |-\frac{1}{8}-\frac{1}{8}| = \frac{1}{4}$$

$$f_{V,W}(v,w) = \frac{1}{4}\frac{1}{\theta^2}\mathrm{e}^{-(v+w)/2-(v-w)/4} = \frac{1}{4}\frac{1}{\theta^2}\mathrm{e}^{-\frac{3}{4}v-\frac{1}{4}w}, \qquad v+w>0,\ v-w>0$$

(b)

$$\begin{aligned} M_{V,W}(s,t) &= E[e^{sV+tW}] = E[e^{s(X+2Y)+t(X-2Y)}] \\ &= E[e^{(s+t)X+(2s-2t)Y}] = M_X(s+t)M_Y(2s-2t) \\ &= \frac{1}{1-\theta(s+t)}\frac{1}{1-\theta(2s-st)} \end{aligned}$$

Omdat dit niet te schrijven is als een product van een functie van s en van t zijn Y en Z niet s.o. Bij (a) had men overigens ook kunnen zien dat de drager geen Cartesisch product is.

3. De CDF is

$$F_X(x) = \begin{cases} 1 - \frac{1}{x^2}, & \text{voor } x > 1 \\ 0, & \text{elders.} \end{cases}$$

(a)

$$\begin{aligned} P\left[\frac{1}{\sqrt{n}}X_{n:n} \le x\right] &= P\left[X_{n:n} \le \sqrt{n}x\right] \\ &= \left(F_X(\sqrt{n}x)\right)^n \\ &= \left(1-\frac{1}{nx^2}\right)^n \overset{n\to\infty}{\longrightarrow} e^{-\frac{1}{x^2}}, \qquad x>0 \end{aligned}$$

(b)

$$\begin{aligned} P\left[|X_{1:n}-1| \le \epsilon\right] &= P\left[X_{1:n}-1 \le \epsilon\right] \\ &= P\left[X_{1:n} \le 1+\epsilon\right] \\ &= 1-\left(\frac{1}{(1+\epsilon)^2}\right)^n \overset{n\to\infty}{\longrightarrow} 1, \qquad \text{voor vaste } \epsilon>0. \end{aligned}$$

$$\begin{aligned} P\left[n(X_{1:n}-1) \le x\right] &= P\left[X_{1:n} \le 1+\frac{x}{n}\right] \\ &= 1-\left(\frac{1}{\left(1+\frac{x}{n}\right)^2}\right)^n \\ &= 1-\left(1+\frac{x}{n}\right)^{-2n} \overset{n\to\infty}{\longrightarrow} 1-e^{-2x} \end{aligned}$$

Deze limietverdeling is de EXP(1/2) verdeling.

(c) Stel $Y=\frac{1}{X^2}$, dan geldt voor Y

$$\begin{aligned} F_Y(y) &= P[Y\le y] = P[\tfrac{1}{X^2}\le y] = P[X^2 \ge \tfrac{1}{y^2}] \\ &= P[X \ge \tfrac{1}{y}] = 1-F_X(\tfrac{1}{y}) = y, \qquad 0<y<1 \end{aligned}$$

zodat $Y \sim \text{UNIF}(0,1)$, Voor $Y_i=\frac{1}{X_i^2}$, geldt, omdat Y een dalende functie van X is, dat $Y_{n:n}=\frac{1}{X_{1:n}^2}$. Er volgt

$$\begin{aligned} E\left[\frac{1}{X_{1:n}}^2\right] &= E\left[Y_{n:n}\right] = n\int_0^1 y y^{n-1}\mathrm{d}y \\ &= \frac{n}{n+1}\, y^{n+1}\Big|_{y=0}^{1} \\ &= \frac{n}{n+1} \end{aligned}$$

Of via de pdf van $X_{1:n}$:

$$F_{X_{1:n}}(x) = 1-\frac{1}{x^{2n}} = 1-x^{-2n}$$

zodat $f_{X_{1:n}}(x)=2nx^{-2n-1}$, en

$$E\left(\frac{1}{X_{1:n}^2}\right) = \int_1^\infty 2nx^{-2n-3}\,\mathrm{d}x = \frac{2n}{2n+2} = \frac{n}{n+1}.$$

4. (a) Het volgende dient berekend te worden:

$$P\left(\frac{V-W}{V+W} < \frac{1}{2}\right).$$

Er geldt

$$\begin{aligned} P\left[\frac{V-W}{V+W} < \frac{1}{2}\right] &= P\left[\frac{V/W-1}{V/W+1} < \frac{1}{2}\right] \\ &= P\left[V/W - 1 < \frac{1}{2}(V/W+1)\right] \\ &= P\left[\frac{1}{2}V/W < 3/2)\right] \\ &= P\left[V/W < 3)\right] = F_{1,1}(3) \end{aligned}$$

(b) De pdfs van V resp. w zijn gegeven door

$$f_V(v) = \frac{1}{\sqrt{2\pi v}} e^{-v/2}, \quad v > 0, \qquad f_W(w) = \frac{1}{\sqrt{2\pi w}} e^{-w/2}, \quad w > 0.$$

Verder is gegeven dat V en W s.o. zijn. Bekijk de transformatie $U = V + W$, en $Q = \frac{V}{V+W}$. Inverse: $V = UQ$ $W = U - UQ$

$$J = \begin{pmatrix} q & u \\ 1-q & -u \end{pmatrix} \qquad |J| = u.$$

$$\begin{aligned} f_{U,Q}(u,q) &= u f_V(uq) f_W(u(1-q)) \\ &= \frac{2u}{\pi}(uq)^{-\frac{1}{2}}(u(1-q))^{-\frac{1}{2}} e^{-uq/2-u(1-q)/2} \\ &= \frac{2}{\pi}(q(1-q))^{-\frac{1}{2}} e^{-u/2}, \qquad u > 0, \, 0 < q < 1 \end{aligned}$$

De pdf factoriseert en de drager is een Cartesisch product, dus zijn U en Q s.o.