Oplossingen Tentamen Kres 3, 120603, 09.30–12.30

1.

$$f_{X,Y,Z}(x,y,z) = \begin{cases} c(x+y)\mathrm{e}^{-x-y-z}, & \text{voor } x,y,z>0 \\ 0, & \text{elders} \end{cases}$$

(a)

$$\begin{aligned} 1 &= c\int_0^\infty\int_0^\infty\int_0^\infty (x+y)\mathrm{e}^{-x-y-z}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= c\int_0^\infty\int_0^\infty (x+y)\mathrm{e}^{-x-y}\,\mathrm{d}x\,\mathrm{d}y \qquad \text{de integraal over } z \text{ is 1 (EXP(1) pdf)} \\ &= 2c\int_0^\infty\int_0^\infty x\mathrm{e}^{-x-y}\,\mathrm{d}x\,\mathrm{d}y \qquad \text{wegens x,y symmetrie} \\ &= 2c\int_0^\infty x\mathrm{e}^{-x}\,\mathrm{d}x \qquad \text{de integraal is de verwachting van een EXP(1) stochast} \\ &= 2c \end{aligned}$$

Dit geeft $c = \frac{1}{2}$

$$f_{X,Y}(x,y) = \frac{1}{2}\int_0^\infty (x+y)\mathrm{e}^{-x-y-z}\,\mathrm{d}z = \frac{1}{2}(x+y)\mathrm{e}^{-x-y}, \qquad (x,y>0)$$

$$\begin{aligned} f_Y(y) &= \frac{1}{2}\int_0^\infty (x+y)\mathrm{e}^{-x-y}\,\mathrm{d}x \\ &= \frac{1}{2}\int_0^\infty x\mathrm{e}^{-x-y}\,\mathrm{d}x + \frac{1}{2}\int_0^\infty y\mathrm{e}^{-x-y}\,\mathrm{d}x \\ &= \frac{1}{2}(1+y)\mathrm{e}^{-y} \qquad (y>0) \end{aligned}$$

(b) Wegens x,y symmetrie hebben X en Y dezelfde pdf,

$$f_X(x) = (1+x)\mathrm{e}^{-x} \qquad (x>0)$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{(x+y)\mathrm{e}^{-x-y}}{(1+x)\mathrm{e}^{-x}} = \frac{(x+y)\mathrm{e}^{-y}}{(1+x)}, \qquad (y>0)$$

$$\begin{aligned} E[Y|X=x] &= \int_0^\infty \frac{y(x+y)\mathrm{e}^{-y}}{(1+x)}\,\mathrm{d}y \\ &= \frac{x}{1+x}\int_0^\infty y\mathrm{e}^{-y}\,\mathrm{d}y + \frac{1}{1+x}\int_0^\infty y^2\mathrm{e}^{-y}\,\mathrm{d}y \\ &= \frac{x+2}{x+1} \end{aligned}$$

De covariantie:

$$E[XY] = E[XE[Y|X]] = \int_0^\infty x\left(\frac{x+2}{x+1}\right)\frac{1}{2}(1+x)e^{-x}\mathrm{d}x = \frac{1}{2}\int_0^\infty (x^2+2x)e^{-x}\,\mathrm{d}x = 2$$

$$E[X] = E[Y] = \frac{1}{2}\int_0^\infty y(1+y)e^{-y}\,\mathrm{d}y = \frac{1}{2}+1 = 1\frac{1}{2}$$

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y] = 2 - (1\frac{1}{2})^2 = -\frac{1}{4}$$

(c) Bepaal de joint pdf van $V = X - Y$ en $W = X + Y$. Zijn V en W s.o? $V = X - Y$, $W = X + Y \Rightarrow X = (V + W)/2$, $Y = (W - V)/2$.

$$J = \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad ||J|| = |\frac{1}{2}| = \frac{1}{2}$$

$$f_{V,W}(v, w) = \frac{1}{4} w \mathrm{e}^{-w}, \qquad v + w > 0,\ w > v$$

V en W zijn niet s.o. (drager)

2. (a) $Y = X_1 + 2X_2$ en $Z = X_1 - 2X_2$, zodat $X_1 = (Y + Z)/2$, $X_2 = (Y - Z)/4$

$$J = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{-1}{4} \end{pmatrix} \qquad ||J|| = |-\frac{1}{8} - \frac{1}{8}| = \frac{1}{4}$$

$$f_{Y,Z}(y, z) = \frac{1}{4}\frac{1}{\theta^2}\mathrm{e}^{-(y+z)/2-(y-z)/4} = \frac{1}{4}\frac{1}{\theta^2}\mathrm{e}^{-\frac{3}{4}y-\frac{1}{4}z}, \qquad y + z > 0,\ y - z > 0$$

(b)

$$\begin{aligned} M_{Y,Z}(s,t) &= E[e^{sY+tZ}] = E[e^{s(X_1+2X_2)+t(x_1-2X_2)}] \\ &= E[e^{(s+t)X_1+(2s-2t)X_2}] = M_{X_1}(s+t)M_{X_2}(2s-2t) \\ &= \frac{1}{1-\theta(s+t)}\frac{1}{1-\theta(2s-st)} \end{aligned}$$

Omdat dit niet te schrijven is als een product van een functie van s en van t zijn Y en Z niet s.o.

3. De CDF is

$$F_X(x) = \begin{cases} 1 - \frac{1}{x^2}, & \text{voor } x > 1 \\ 0, & \text{elders.} \end{cases}$$

(a)

$$\begin{aligned} P\left[\frac{1}{\sqrt{n}}X_{n:n} \le x\right] &= P\left[X_{n:n} \le \sqrt{n}x\right] \\ &= \left(F_X(\sqrt{n}x)\right)^n \\ &= \left(1 - \frac{1}{nx^2}\right)^n \stackrel{n\to\infty}{\longrightarrow} e^{-\frac{1}{x^2}}, \qquad x > 0 \end{aligned}$$

(b)

$$\begin{aligned} P\left[|X_{1:n} - 1| \le \epsilon\right] &= P\left[X_{1:n} - 1 \le \epsilon\right] \\ &= P\left[X_{1:n} \le 1 + \epsilon\right] \\ &= 1 - \left(\frac{1}{(1+\epsilon)^2}\right)^n \stackrel{n\to\infty}{\longrightarrow} 1, \qquad \text{voor vaste } \epsilon > 0. \end{aligned}$$

$$
\begin{aligned}
P\left[n(X_{1:n}-1) \le x\right] &= P\left[X_{1:n} \le 1+\frac{x}{n}\right]\\
&= 1-\left(\frac{1}{\left(1+\frac{x}{n}\right)^2}\right)^n\\
&= 1-\left(1+\frac{x}{n}\right)^{-2n} \stackrel{n\to\infty}{\longrightarrow} 1-e^{-2x}
\end{aligned}
$$

Deze limietverdeling is de EXP(1/2) verdeling.

(c) Stel $Y=\frac{1}{X^2}$, dan geldt voor Y

$$
\begin{aligned}
F_Y(y) &= P[Y\le y] = P[\tfrac{1}{X^2}\le y] = P[X^2 \ge \tfrac{1}{y^2}]\\
&= P[X\ge \tfrac{1}{y}] = 1-F_X(\tfrac{1}{y}) = y, \qquad 0<y<1
\end{aligned}
$$

zodat $Y\sim$ UNIF$(0,1)$, Voor $Y_i=\frac{1}{X_i^2}$, geldt, omdat Y een dalende functie van X is, dat $Y_{1:n}=\frac{1}{X_{n:n}^2}$. Er volgt

$$
E\left[\frac{1}{X_{n:n}}^2\right] = E\left[Y_{1:n}\right] = \int_0^1 yy^{n-1}\mathrm{d}y = \frac{1}{n+1}y^n|_{y=0}^1 = \frac{1}{n+1}
$$

4. (a) Het volgende dient berekend te worden:

$$
P\left[\frac{X_1^2-X_2^2}{X_1^2+X_2^2}<\frac{1}{2}\right].
$$

Laat $Z_i=X_i^2$, voor $i=1,2$, dan

$$
\begin{aligned}
P\left[\frac{X_1^2-X_2^2}{X_1^2+X_2^2}<\frac{1}{2}\right] &= P\left[\frac{Z_1-Z_2}{Z_1+Z_2}<\frac{1}{2}\right]\\
&= P\left[\frac{Z_1/Z_2-1}{Z_1/Z_2+1}<\frac{1}{2}\right]\\
&= P\left[Z_1/Z_2-1<\frac{1}{2}(Z_1/Z_2+1)\right]\\
&= P\left[\frac{1}{2}Z_1/Z_2<3/2)\right]\\
&= P\left[Z_1/Z_2<3)\right] = F_{1,1}(3)
\end{aligned}
$$

(b) Stel wederom $Z_i=X_i^2$. De pdf van Z is gegeven door

$$
f_Z(z) = \frac{2}{\sqrt{2\pi z}}e^{-z/2}.
$$

Bekijk de transformatie $U=X_1^2+X_2^2=Z_1+Z_2$, en $V=\frac{X_1^2}{X_1^2+X_2^2}=\frac{Z_1}{Z_1+Z_2}$ $Z_1=UV$ $Z_2=U-UV$

$$
J=\begin{pmatrix} v & u\\ 1-v & -u\end{pmatrix} \qquad ||J||=u.
$$

$$
\begin{aligned}
f_{U,V}(u,v) &= u f_Z(uv) f_Z(u(1-v)) \\
&= \frac{2u}{\pi}(uv)^{-\frac{1}{2}}(u(1-v))^{-\frac{1}{2}} e^{-uv/2-u(1-v)/2} \\
&= \frac{2u}{\pi} u^{-1}(v(1-v))^{-\frac{1}{2}} e^{-u/2}, \qquad u>0,\ 0<v<1
\end{aligned}
$$