Oplossingen Tentamen Kres 4, 220304

1. (a) REC $\Rightarrow S := \sum X_i$ complete en sufficient
$X_i \sim \text{GAM}(\theta^{-1}, 7/5) \Rightarrow S \sim \text{GAM}(\theta^{-1}, 7n/5)$

$$f_S(s) = \frac{\theta^{7n/5} s^{7n/5-1} e^{-\theta s}}{\Gamma(7n/5)}, \qquad s > 0.$$

$$E\frac{1}{S} = \int_0^\infty \frac{\theta^{7n/5} s^{7n/5-2} e^{-\theta s}}{\Gamma(7n/5)} = \frac{\theta\Gamma(7n/5-1)}{\Gamma(7n/5)} = \frac{\theta}{7n/5-1}$$

$\Rightarrow \frac{7n-5}{5S}$ is UMVUE voor θ

(b) $\log f(x;\theta) = c + \frac{7}{5}\log\theta - \theta x$

$$\frac{d^2 \log f(x;\theta)}{d\theta^2} = -\frac{7}{5\theta^2} \qquad \Rightarrow$$

$\text{CRLB} = \frac{\left(\frac{d}{d\theta}\theta^{-2}\right)^2}{\frac{7}{5}\frac{n}{\theta^2}} = \frac{(-2\theta^{-3})^2\theta^2}{\frac{7}{5}n} = \frac{4\theta^{-4}}{\frac{7}{5}n} = \frac{20}{7\theta^4}$

(c)

$$\frac{2^{7n/5}\prod x_i^{2/5} e^{-2\sum x_i}}{3^{7n/5}\prod x_i^{2/5} e^{-3\sum x_i}} = \left(\frac{2}{3}\right)^{7n/5} e^{\sum x_i} < k$$

Verwerpen als $\sum x_i < k^*$, met $2 \times 2S|\theta = 2 \sim \text{GAM}(\frac{1}{2}, 14n/5)$, ofwel $4\sum X_i|\theta = 2 \sim \chi^2(14n/5)$

$\Rightarrow$ Verwerp als $\sum x_i < \frac{\chi^2(14n/5)}{4}$

(d)

$$\lambda = \begin{cases} 1 & \text{als } \frac{7n}{5s} \leq 2 \\ \left(\frac{10s}{7n}\right)^{7n/5} e^{-2s+7n/5} & \text{als } \frac{7n}{5s} > 2 \end{cases}$$

λ is niet-dalend in s omdat

$$\frac{d\lambda}{ds} = \left(\frac{14n}{10s} - 2\right)\left(\frac{10s}{7n}\right)^{7n/5} e^{-2s+7n/5} > 0 \qquad \text{voor } \frac{7n}{5s} > 2.$$

Verwerpen als $s \leq k$ met

$$P[S < k|\theta = 2] = \alpha.$$

Omdat $2 \times 2S|\theta = 2 \sim \chi^2(14n/5)$, is

$$k = \chi^2_\alpha(14n/5).$$

(e) $2\theta S \sim \chi^2(14n/5)$,

$$\text{BI} = \left(\frac{\chi^2_{\frac{1-\gamma}{2}}\left(\frac{14n}{5}\right)}{2S}, \frac{\chi^2_{\frac{1+\gamma}{2}}\left(\frac{14n}{5}\right)}{2S}\right).$$

2. (a) $L(\theta) = \frac{13^n \theta^{13n}}{\prod x_i^4} I_{[\theta,\infty)}(x_{1:n}) \Rightarrow X_{1:n}$ MLE want dan $L(\theta)$ maximaal.

$$f_{X_{1:n}(x)} = \frac{13n\theta^{13}}{x^{14}} \left(\int_x^\infty \frac{13\theta^{13}}{t^{14}} \mathrm{d}t \right)^{n-1} = \frac{13n\theta^{13}}{x^{14}} \left(\frac{\theta^{13}}{x^{13}} \right)^{n-1} = \frac{13n\theta^{13n}}{x^{13n+1}} \qquad (x > \theta)$$

$$E(X_{1:n}^k) = \int_\theta^\infty \frac{13n\theta^{13n}}{x^{13n+1-k}} \mathrm{d}x = -\frac{13n\theta^{13n}}{(13n-k)} x^{k-13n} \,|_\theta^\infty = \frac{13n}{13n-k} \theta^k$$

$$\mathrm{MSE}(X_{1:n}) = \theta^2 \left\{ \frac{13n}{13n-2} - \left(\frac{13n}{13n-1} \right)^2 \right\} + \theta^2 \left(\frac{13n}{13n-1} - 1 \right)^2$$

MSE consistentie volgt uit: $\lim_{n\to\infty} \mathrm{MSE}(X_{1:n}) = 0$

(b) Uit 2a volgt $X_{1:n}$ sufficient. Eis

$$Eu(X_{1:n}) = \int_\theta^\infty \frac{u(x) 13n\theta^{13n}}{x^{13n+1}} \mathrm{d}x = 0 \qquad \text{ofwel} \qquad \int_\theta^\infty \frac{u(x)}{x^{13n+1}} \mathrm{d}x = 0.$$

Afgeleide naar θ nemen geeft: $-\frac{u(\theta)}{\theta^{13n+1}} = 0$ wat $u(\theta) = 0$ impliceert. Hieruit volgt dat $X_{1:n}$ tevens complete is.

$$E\left(x_{1:n}^{-4}\right) = \int_\theta^\infty \frac{13n\theta^{13n}}{x^{13n+1+4}} \mathrm{d}x = \frac{13n}{4-13n} \frac{\theta^{13n}}{x^{13n+4}} = \frac{13n}{13n+4} \theta^{-4}$$

$\Rightarrow \frac{13n+4}{13n} X_{1:n}^{-4}$ UMVUE voor θ^{-4}

(c) voor $\theta_2 > \theta_1$:

$$\frac{\theta_2^{13n} \prod x_i}{\theta_1^{13n} \prod x_i} \frac{I_{[\theta_2,\infty)}(x_{1:n})}{I_{[\theta_1,\infty)}(x_{1:n})} = \left(\frac{\theta_2}{\theta_1} \right)^{13n} \begin{cases} 0 & \theta_1 \le x_{1:n} < \theta_2 \\ 1 & x_{1:n} \ge \theta_2 \end{cases}$$

monotoon niet-dalend in $x_{1:n}$. Verwerp als $x_{1:n} > k$ met $P[X_{1:n} > k | \theta = 10] = \int_k^\infty \frac{13n 10^{13n}}{x^{13n+1}} \mathrm{d}x = \frac{10^{13n}}{k^{13n}} = \alpha \Rightarrow k = \frac{10}{\alpha^{1/(13n)}}$

(d) $Z := \frac{X}{\theta} \Rightarrow \frac{dX}{dZ} = \theta \Rightarrow f_Z(z) = \frac{\theta 13\theta^{13n}}{\theta^{14} z^{14}} = \frac{13}{z^{14}} I_{[1,\infty)}(z)$
$f_Z(z)$ is onafhankelijk van $\theta \Rightarrow$

$$\frac{X_{n:n} - X_{1:n}}{X_{1:n}} = \frac{Z_{n:n} - Z_{1:n}}{Z_{1:n}}$$

heeft een verdeling die niet van θ afhangt dus volgens de stelling van Basu volgt onafhankelijkheid van de complete en sufficient statistic $X_{1:n}$

(e) $Q := \frac{X_{1:n}}{\theta} \Rightarrow X_{1:n} = \theta Q \Rightarrow \frac{dX_{1:n}}{dQ} = \theta \Rightarrow f_Q(q) = \frac{13n}{q^{13n+1}} I_{(1,\infty)}(q)$
We zoeken q_1 en q_2 zodanig dat

$$\int_1^{q_1} f_Q(q) \mathrm{d}q = \frac{1-\gamma}{2}$$

en

$$\int_{q_2}^{\infty} f_Q(q)\mathrm{d}q = \frac{1-\gamma}{2}$$

Voor q_1 geeft dit: $1 - \frac{1}{q^{13n}} = \frac{1-\gamma}{2}$, met als oplossing $q_1 = \left(\frac{2}{1+\gamma}\right)^{1/(13n)}$. Op soortgelijke wijze: $q_2 = \left(\frac{2}{1-\gamma}\right)^{1/(13n)}$. $P[q_1 \leq \frac{X_{1:n}}{\theta} \leq q_2] = \gamma$, zodat:

$$\mathrm{BI} = \left(\frac{q_2}{X_{1:n}}, \frac{q_1}{X_{1:n}}\right)$$