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SOME SOLUTIONS

1. (a) To express the GARCH(1,2) process as an infinite order ARCH process,

$$\begin{aligned}(1-\beta_1 L)\sigma_t^2 &= \omega + \alpha_1 y_{t-1}^2 + \alpha_2 y_{t-2}^2\\ \sigma_t^2 &= (1-\beta_1 L)^{-1}(\omega + \alpha_1 y_{t-1}^2 + \alpha_2 y_{t-2}^2)\\ \sigma_t^2 &= \frac{\omega}{1-\beta_1 L} + \frac{(\alpha_1+\alpha_2 L)y_{t-1}^2}{1-\beta_1 L}.\end{aligned}$$

On collecting terms of

$$(1-\beta_1 L)^{-1}(\alpha_1+\alpha_2 L) = (1+\beta_1 L + \beta_1^2 L^2 + \ldots)(\alpha_1+\alpha_2 L)$$

$$\sigma_t^2 = \frac{\omega}{1-\beta_1} + \Big[\alpha_1 + (\alpha_1\beta_1+\alpha_2)\sum_{j=1}^{\infty}\beta_1^{j-1}L^j\Big]y_t^2$$

so there is exponential decay in the impulse response weights of the infinite order ARCH representation for the GARCH(1,2) process.

(b) The non-negativity conditions for the process are $\omega > 0$, $\alpha_1 \geqslant 0$, $\alpha_2 \geqslant 0$, $\beta_1 \geqslant 0$.

(c) The unconditional variance for the returns series is,

$$Var(y_t) = \Big(\frac{\omega}{1-\alpha_1-\alpha_2-\beta_1}\Big),$$

while the autocovariance of returns at lag k is

$$\gamma_k = E(y_t y_{t-k}) = E(z_t\sigma_t z_{t-k}\sigma_{t-k}) = E\{(z_t z_{t-k})(\sigma_t\sigma_{t-k})\} = 0$$

since $z_t \sim i.i.d.(0,1)$. Hence returns are uncorrelated.

(d) For y_t to be covariance stationary it is necessary for it to have a finite variance in addition to the fact that its autocovariance function is trivially defined above. From part (c) it then follows that $0 < (\alpha_1+\alpha_2+\beta_1) < 1$.

(e) For the derivation of the ARMA(p,q) representation of a GARCH(1,2) model it is necessary to introduce squared returns to both sides of the equation and to re-arrange as,

$$\begin{aligned}y_t^2 &= y_t^2 - \sigma_t^2 + \omega + \alpha_1 y_{t-1}^2 + \alpha_2 y_{t-2}^2 + \beta_1\sigma_{t-1}^2\\ y_t^2 &= \omega + (y_t^2-\sigma_t^2) + (\alpha_1+\beta_1)y_{t-1}^2 - \beta_1(y_{t-1}^2-\sigma_{t-1}^2) + \alpha_2 y_{t-2}^2\\ y_t^2 &= \omega + (\alpha_1+\beta_1)y_{t-1}^2 + \alpha_2 y_{t-2}^2 + \nu_t - \beta_1\nu_{t-1}\end{aligned}$$

where $\nu_t = (y_t^2 - \sigma_t^2)$ and is a white noise process. Hence y_t^2 is an ARMA(2,1) process.

(f) The theoretical population autocorrelation function of y_t^2 must be obtained for

$$[1-(\alpha_1+\beta_1)L - \alpha_2 L^2]y_t^2 = \omega + (1-\beta_1 L)\nu_t.$$

Note that the autoregressive part of the model is obtained from the solution of the homogenous difference equation, $z^2 - (\alpha_1+\beta_1)z - \alpha_2 = 0$. Since $(\alpha_1+\beta_1)^2 + 4\alpha_2 > 0$, then the roots will be real and the autocorrelation function will be the sum of two exponentially decaying terms after lag one.

2. (a) $y_t = y_{t-1} - (2/9)y_{t-2} + \varepsilon_t$ is an AR(2) process. To check whether or not the process is stationary, we need to examine the lag polynomial, $\phi(L) = 1 - L + (2/9)L^2$. The auxilliary equation is

$$\Big(1 - z + (2/9)z^2\Big) = \Big(1-\frac{z}{3}\Big)\Big(1-\frac{2z}{3}\Big) = 0,$$

as factors, which implies roots of $z_1 = 3$ and $z_2 = 3/2$. Since both roots lie outside the unit circle, the process is stationary.

(b) For the Wold Decomposition $y_t = \Psi(L)\varepsilon_t = \sum_{k=0}^{\infty}\Psi_k\varepsilon_{t-k}$

$$\Big(1 - L + \frac{2}{9}L^2\Big)(1 + \Psi_1 L + \Psi_2 L^2 + \ldots + \Psi_k L^k + \ldots) = 1.$$

Hence, $\phi(L)\Psi_k = 0$ $(k \geq 1)$, $\Psi_k = A(1/z_1)^k + B(1/z_2)^k$ and $\Psi_k = A(1/3)^k + B(2/3)^k$. But $\phi_0 = 1$ and from successive substitution,

$$\begin{aligned}y_t &= \varepsilon_t + y_{t-1} - (2/9)y_{t-2} = \varepsilon_t + (\varepsilon_{t-1} + y_{t-2} - (2/9)y_{t-3}) - (2/9)y_{t-2}\\ &= \varepsilon_t + \varepsilon_{t-1} + (7/9)y_{t-2} - (2/9)y_{t-3}.\end{aligned}$$

So, $\Psi_0 = 1$ and $\Psi_1 = 1$ which are enough to solve for A and B. If we were to continue with the successive substitution we would find that $\Psi_0 = 1$, $\Psi_1 = 1$, $\Psi_2 = (7/9)$, $\Psi_3 = (5/9)$, $\Psi_4 = (31/81), \ldots$. Anyway, just using the first two gives, $\Psi_0 = 1 = A + B$, $\Psi_1 = 1 = (1/3)A + (2/3)B$ which gives $B = 2$ and 1 $A = -1$; therefore

$$\Psi_k = -\Big(\frac{1}{3}\Big)^k + 2\Big(\frac{2}{3}\Big)^k.$$

(c) Use the Yule Walker equations for $y_t - y_{t-1} + (2/9)y_{t-2} = \varepsilon_t$; hence the first two for lags 1 and 2 are:

$$\rho_1 - 1 + (2/9)\rho_1 = 0 \quad\text{and}\quad \rho_2 = \rho_1 - (2/9)\rho_0 = \rho_1 - (2/9)$$

since $\rho_0 = 1$, then it follows form the first Yule Walker equation that $\rho_1 = (9/11)$. Using these two autocorrelations to eliminate the A and B constants gives,

$$\rho_k = -(5/11)\Big(\frac{1}{3}\Big)^k + (16/11)\Big(\frac{2}{3}\Big)^k \quad k \geq 0.$$

(d) This ARMA(2,1) model can be expressed as

$$\Big(1 - L + (2/9)L^2\Big)y_t = \Big(1-(L/3)\Big)\Big(1-(2L/3)\Big)y_t = \Big(1-(L/3)\Big)\varepsilon_t.$$

Hence there is a common factor and the model reduces to an AR(1) process. To work with the original ARMA(2,1) model would be problematic, since it is *not identified* and attempt to estimate it would lead to unstable parameter estimates, and trouble doing any inference.

3. (a) By definition $\gamma_k = Cov(y_t, y_{t-k}) = E[y_t - E(y_t)][y_{t-k} - E(y_{t-k})]$. And for the trend stationary process $\gamma_k = E[\alpha + \beta t + u_t - E(y_t)][\alpha + \beta(t-k) + u_{t-k} - E(y_{t-k})]$. Hence $\gamma_k = E(u_t u_{t-k})$ which is well defined and just the usual autocovariance function for an MA(1) process, with $\gamma_0 = \sigma^2(1+\theta^2)$, $\gamma_1 = -\theta\sigma^2$, and $\gamma_k =$ for $k \geqslant 2$. So, no problem!

5 a. Because $\{\varepsilon_{1i}\}$ and $\{\varepsilon_{2i}\}$ are 2 sequences of i.i.d. random variables, they are also white noise. The reason is that $E(\varepsilon_{jt}) = u_j$ = independent of t $Var(\varepsilon_{jt}) = \sigma_j^2$ = independent of t and $Cov(\varepsilon_{jt}, \varepsilon_{js}) = \sigma^2$ for $t = s$ and 0 for $t \neq s$. Accordingly $X_{2t} = \varepsilon_{2t}$ is $I(0)$.

b. From $\Delta X_{1t} = X_{1t} - X_{1,t-1} = \varepsilon_{1t}$, we know that it is a white noise process. Hence, by definition, X_{1t} is $I(1)$.

c. It can be shown that

$$\Delta X_t = X_t - X_{t-1} = \varepsilon_{1t} + (\varepsilon_{2t} - \varepsilon_{2,t-1}),$$

which is a stationary process. The reason is that

(a) $E(\Delta X_t) = E(\varepsilon_{1t})$ = constant.

(b) $Var(\Delta X_t) = Var(\varepsilon_{1t} + (\varepsilon_{2t} - \varepsilon_{2,t-1})) = \sigma_1^2 + 2\sigma_2^2$ = constant;

(c) For $s < t$, it holds that

$$\begin{aligned} Cov(\Delta X_s, \Delta X_t) &= Cov(\varepsilon_{1s} + (\varepsilon_{2s} - \varepsilon_{2,s-1}), \varepsilon_{1t} + (\varepsilon_{2t} - \varepsilon_{2,t-1})), \\ &= Cov(\varepsilon_{1s}, \varepsilon_{1t}) + Cov(\varepsilon_{2s}, \varepsilon_{2t}) - Cov(\varepsilon_{2s}, \varepsilon_{2,t-1}) \\ &\quad -Cov(\varepsilon_{2,s-1}, \varepsilon_{2t}) + Cov(\varepsilon_{2,s-1}, \varepsilon_{2,t-1}) \\ &= \begin{cases} \sigma_1^2 + 2\sigma_2^2, & \text{if } s = t; \\ -\sigma_2^2, & \text{if } |s-t| = 1; \\ 0, & \text{if } |s-t| > 1. \end{cases} \end{aligned}$$

Hence, X_t is $I(1)$.

6a.+b. The two equations can be written as a VAR model for the variables y_t and x_t, using the notation $Y_t = (y_t, x_t)'$, like

$$Y_t = A_1 Y_{t-1} + A_2 Y_{t-2} + \varepsilon_t$$

with

$$A_1 = \begin{bmatrix} 1+\gamma+\alpha & -\gamma \\ \beta & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} -\alpha & 0 \\ -\beta & 0 \end{bmatrix}$$

and $\varepsilon_t = (v_t, w_t)'$. Note

$$\begin{aligned} \Delta Y_t &= A_1 Y_{t-1} + A_2 Y_{t-2} - Y_{t-1} + \varepsilon_t \\ &= \prod Y_{t-1} - A_2 Y_{t-1} + A_2 Y_{t-2} + \varepsilon_t \\ &= \prod Y_{t-1} - A_2 \Delta Y_{t-1} + \varepsilon_t \end{aligned}$$

where $\prod = A_1 + A_2 - I_2$. Hence

$$\prod = \begin{bmatrix} \gamma & -\gamma \\ 0 & 0 \end{bmatrix}$$

and hence it has rank 1. Thus the two series are cointegrated of the first order.