Master of Econometrics - Game Theory

08-01-2010, from 14.00 till 17.00 hours.

Exercise 1 A correct and well motivated answer to each of the following problems scores 15 points.

a *Auctions*

Explain and justify the following claims.

- A *first* price sealed bid auction is equivalent with a *descending* (Dutch) auction.
- A *second* price sealed bid auction is related to but not quite equivalent with an *ascending* sequential (English) auction. Explain the essential differences.
- In *public value* auctions participants tend to bid too much (the *winner's curse*).

b *Equilibria of Strategic Games*

Let G be a finite strategic game with the set $A = A_1 \times ... \times A_n$ of action profiles.

- Define (in words or formulas) a *strategic* equilibrium of G in mixed strategies.
- Sketch the idea of the proof of existence of such an equilibrium (Nash).
- Define (in words or formulas) a *correlated* equilibrium of G (Aumann).
  You may use A as the space of states.

c *Sequential and Trembling Hand Perfect Equilibria*

Let Γ be a finite extensive game with *imperfect* information.

- Give a precise definition of a *weakly sequential* equilibrium of Γ.
- A *sequential* equilibrium must have an additional property. What is it?
- Explain how a *trembling hand perfect* equilibrium can be used in order to prove that such a sequential equilibrium always exists.

d *Signaling and Screening*

Let Γ be a signaling game with n types of the sender who can send one of m signals, and k actions for the receiver, each of which can be played after every signal.

- How many pure strategies have both players in the strategic form of Γ?
- How many players are there in the agent strategic form of Γ?

In the corresponding *screening* game the sender commits himself in advance openly to one of his, possible mixed, strategies available to him in the signaling game.

- Describe, in words or symbols, the spaces of strategies of both players in the screening game with the same types, signals and actions as in Γ.

Exercise 2 (30) *Auctions*

a Formulate the Revenue Equivalence Theorem in its most general form (known to you).

Consider two risk averse bidders with the vNM utility function $u(x) = x^c,\ 0 < c < 1$ and private valuations distributed independently and uniformly on $[0, 1]$.

b What auction format, the first price or the second price, would you choose as a risk neutral seller? (reason by a comparison with the case of risk neutral bidders).

c Derive the differential equation for the symmetric equilibrium bidding function $b(v)$ in the case of two risk averse bidders. Show that the solutions of this equation are

$$b(v) = \frac{v}{1+c} + \frac{A}{v^{1/c}}, \quad \text{with some constant } A \geq 0.$$

d The seller can increase her expected revenue by excluding bidders with low valuations.

Determine a as a function of the imposed reservation price r.

e The value of r maximizing the expected profit of the seller will depend on c.

Should you raise or lower r as the bidders are becoming increasingly risk averse?

f Calculate the optimal value of r for the case of the risk neutral bidders, $c = 1$,

and that for the case of the risk averse bidders with $c = \frac{1}{2}$.

Solution 2

b With risk neutral bidders both formats, first and the second price, provide the same expected revenue for the seller. Hence, a risk neutral seller has no obvious reason to prefer one above the other. However the equilibrium distribution of prices will be diffrent. One can prove that a risk averse seller is better off with a first price auction.

In the equilibrium of a second price auction risk averse buyers still bid their values, while in the equlibrium of a first price auction they should bid more than risk neutral buyers. Hence, in this case also a risk neutral seller will prefer a first price auction.

The same argument holds for auctions of both kinds with the same reservation price.

c The expected payoff of buyer i with value v_i who bids b_i and believes that the other buyer bids according to the bidding function $b_j = b(v_j)$ is

$$E\pi_i(v_i,\ b_i \mid b) = b^{-1}(b_i)(v_i - b_i)^c.$$

The optimal bid b_i must satisfy the first order condition

$$\frac{\partial E\pi_i(v_i,\ b_i \mid b)}{\partial b_i} = \left(b'\left(b^{-1}(b_i)\right)\right)^{-1}(v_i - b_i)^c - c\left(b^{-1}(b_i)\right)(v_i - b_i)^{c-1} = 0.$$

In a symmetric equilibrium we can substitute $b_i = b(v_i)$. Simpifying and dropping the subscripts we obtain a differential equation for the bidding function

$$v - b(v) = cvb'(v).$$

bonus The subtitution $y = vb^c$ allows us to separate the variables and to write

$$y^{(1-c)/c}dy = v^{1/c}dv.$$

This equation has solutions of the form

$$y^{1/c} = \frac{1}{1+c}v^{(1+1/c)} + A,$$

so that eventually

$$b = y^{1/c}v^{-1/c} = \frac{v}{1+c} + \frac{A}{v^{1/c}}, \quad \text{with some constant } A \geq 0.$$

d The reservation price of r implies that buyers with values less than r will not bid, and that a buyer with the value r will bid her value, so that her expected payoff is 0. Hence

$$b(r) = \frac{r}{1+c} + \frac{A}{r^{1/c}} = r, \text{ which implies that } A = \frac{cr^{(1+1/c)}}{1+c}.$$

Finally, for the first price action two risk averse bidders and the reservation price r we find the equilibrium bidding funcion

$$b(v) = \frac{1}{1+c}\left(v + cr\left(\frac{r}{v}\right)^{1/c}\right).$$

e We expect the profit maximizing reservation price $r^*(c)$ to be an increasing function of c. For relatively low values of c we do not expect to gain much from excluding buyers with low valuations as risk averse buyers with high valuations are expected to bid anyhow more that their less risky counterparts.

f The optimal reservation price can be computed exactly for all values of $0 < c \leq 1$.

The expected profit of the seller for given r and c can be evaluated as

$$E\pi(r,c) = 2\int_r^1 vb(v)\,\mathrm{d}v = \frac{2}{1+c}\int_r^1 v^2 + cr^{(1+1/c)}v^{(1-1/c)}\mathrm{d}v.$$

The profit maximizing reservation price $r^*(c)$ satisfies the first order condition

$$\begin{aligned}\frac{\partial E\pi(r,c)}{\partial r} &= \frac{2}{1+c}\left(\int_r^1 (1+c)\,r^{1/c}v^{(1-1/c)}\mathrm{d}v - \left(r^2 + cr^2\right)\right) = 2\left(\frac{c}{2c-1}\left(r^{1/c} - r^2\right) - r^2\right) \\ &= \frac{2}{2c-1}\left(cr^{1/c} - (3c-1)\,r^2\right) = 0, \quad \text{for } \frac{1}{3} \leq c \leq 1.\end{aligned}$$

For $c < \frac{1}{3}$ the derivative $\partial E\pi(r,c)/\partial r$ is negative for all $0 \leq r \leq 1$. Hence

$$r^*(c) = \begin{cases} 0 & \text{for } 0 < c \leq \frac{1}{3}, \\ \left(3 - \frac{1}{c}\right)^{c/(1-2c)} & \text{for } \frac{1}{3} < c < \frac{1}{2}, \\ e^{-1} & \text{for } c = \frac{1}{2}, \\ \left(3 - \frac{1}{c}\right)^{-c/(2c-1)} & \text{for } \frac{1}{2} < c < 1. \end{cases}$$

As expected $r^*(c)$ is indeed a strictly increasing function of c for $\frac{1}{3} \leq c \leq 1$.

Somewhat surprisingly $r^*(c) = 0$ for all c smaller than $\frac{1}{3}$.

Exercise 3 (30) *Signaling Games*

The incumbent firm $\boldsymbol{A}$ is threatened by the entry of firm $\boldsymbol{B}$. The financial situation of $\boldsymbol{A}$ is strong - type $\boldsymbol{S}$, with probability ρ, or weak - type $\boldsymbol{W}$, with probability $1-\rho$.

$\boldsymbol{A}$ can report her situation as good - signal $\boldsymbol{g}$, or bad - signal $\boldsymbol{b}$. $\boldsymbol{B}$ is not able to check if $\boldsymbol{A}$'s report can be trusted. Eventually $\boldsymbol{A}$ will have to pay 1 for a false report.

In any case $\boldsymbol{A}$ earns 2 if $\boldsymbol{B}$ quits - $\boldsymbol{out}$, and loses 1 if $\boldsymbol{B}$ enters - $\boldsymbol{in}$. $\boldsymbol{B}$ earns 0 if he stays $\boldsymbol{out}$. After the entry he gains 1 if $\boldsymbol{A}$ is weak, and loses 1 if $\boldsymbol{A}$ is strong.

$\boldsymbol{A}$ and $\boldsymbol{B}$ are playing a sequential equilibrium of this signaling game.

a Prove that there is no separating equilibrium, for any value of ρ.

b $\rho = \frac{3}{4}$. Construct two pooling equilibria.

Which of them is unreasonable? Provide a plausible argument.

c $\rho = \frac{1}{4}$. Prove that in this case there is no pooling equilibrium either.

Construct a hybrid equilibrium in mixed strategies.

Provide a full specification of this (weakly) sequential equilibrium.

Solution 3

a In a separating equilibrium $\boldsymbol{B}$ enters after the signal send by $\boldsymbol{W}$ and stays out after the signal send by $\boldsymbol{S}$. Then $\boldsymbol{W}$ can profit by sending the same signal as $\boldsymbol{S}$. No equilibrium.

b **Equilibrium I**

$$\text{signals}: \begin{pmatrix} \boldsymbol{S}: \boldsymbol{g} \\ \boldsymbol{W}: \boldsymbol{g} \end{pmatrix}, \qquad \text{beliefs}: \begin{pmatrix} \boldsymbol{g}: \left(\frac{3}{4},\ \frac{1}{4}\right) \\ \boldsymbol{b}: \left(\sigma \le \frac{1}{2},\ \omega \ge \frac{1}{2}\right) \end{pmatrix}, \qquad \text{actions}: \begin{pmatrix} \boldsymbol{g}: \boldsymbol{out} \\ \boldsymbol{b}: \boldsymbol{in} \end{pmatrix}.$$

Equilibrium II

$$\text{signals}: \begin{pmatrix} \boldsymbol{S}: \boldsymbol{b} \\ \boldsymbol{W}: \boldsymbol{b} \end{pmatrix}, \qquad \text{beliefs}: \begin{pmatrix} \boldsymbol{g}: \left(\sigma \le \frac{1}{2},\ \omega \ge \frac{1}{2}\right) \\ \boldsymbol{b}: \left(\frac{3}{4},\ \frac{1}{4}\right) \end{pmatrix}, \qquad \text{actions}: \begin{pmatrix} \boldsymbol{g}: \boldsymbol{in} \\ \boldsymbol{b}: \boldsymbol{out} \end{pmatrix}.$$

In the strategic form second equilibrium is eliminated by iterated weak dominance.

In the extensive form it is unreasonable for $\boldsymbol{B}$ to assume after an unexpected good report that it comes from the weak type of $\boldsymbol{A}$ rather than from the strong type of $\boldsymbol{A}$.

c In a pooling equilibrium in pure strategies firm $\boldsymbol{B}$ will enter after both signals if $\rho < \frac{1}{2}$. But then $\boldsymbol{S}$ will signal $\boldsymbol{g}$ and $\boldsymbol{W}$ will signal $\boldsymbol{b}$. No equilibrium.

Equilibrium III

$$\text{signals}: \begin{pmatrix} \boldsymbol{S}: \boldsymbol{g} \\ \boldsymbol{W}: \left(\gamma = \frac{1}{3},\ \beta = \frac{2}{3}\right) \end{pmatrix}, \ \text{beliefs}: \begin{pmatrix} \boldsymbol{g}: \left(\sigma = \frac{1}{2},\ \omega = \frac{1}{2}\right) \\ \boldsymbol{b}: (0,\ 1) \end{pmatrix}, \ \text{actions}: \begin{pmatrix} \boldsymbol{g}: \left(\iota = \frac{2}{3},\ o = \frac{1}{3}\right) \\ \boldsymbol{b}: \boldsymbol{in} \end{pmatrix}.$$

At this equilibrium $\boldsymbol{W}$ can randomize because she is indifferent between $\boldsymbol{g}$ and $\boldsymbol{b}$, due to $\boldsymbol{B}$'s actions,.so as to make $\boldsymbol{B}$ indiffrent between $\boldsymbol{in}$ and $\boldsymbol{out}$ after $\boldsymbol{g}$.

$\boldsymbol{B}$ beliefs after $\boldsymbol{g}$ that both types are equally likely which makes him indifferent between $\boldsymbol{in}$ and $\boldsymbol{out}$ and allows him to randomize so as to make $\boldsymbol{W}$ indifferent between $\boldsymbol{g}$ and $\boldsymbol{b}$.